Some propositions about the use of Ornstein-Uhlenbeck process for degradation modeling and RUL estimation

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Introduction & Outline

Context & Motivations

✗ **Context:**

- Management of deteriorating systems or components
- Gradually deteriorating systems or devices
- Degradation records at (random) discrete inspection times
- Failure time = first passage time

✗ **Motivation:** built a stochastic process for degradation modeling under the assumption that information is given about mean and variance of the degradation evolution

➩ Lifetime prognostic (Remaining Useful Lifetime)

➩ Extend the list of stochastic processes that can help for maintenance decision making and optimisation

Outline

- ✗ Time dependent Ornstein-Uhlenbeck Process
- ✗ Parametric Estimation
- ✗ Remaining Useful Lifetime Estimation
- ✗ Wiener with drift vs time dependent O.-U.
- ✗ Conclusion

Time dependent Ornstein-Uhlenbeck Process

Ornstein-Uhlenbeck Process

- **Definition**: Stochastic Process solution of the SDE

$$X_t = X_0 + \int_0^t \lambda(\mu - X_u)\,du + \int_0^t \sigma\,dB_u$$

where $\mu, \lambda > 0$ and $\sigma > 0$ are parameters and B_t is the standard Brownian motion.

- Explicit solution

$$X_t = x_0 e^{-\lambda t} + \mu(1 - e^{-\lambda t}) + \sigma \int_0^t e^{-\lambda(t-s)}\,dB_s$$

- Stationary, Gaussian and Markovian process.
- Mean reverting property: over time, X_t tends to drift toward its long-term mean.

Time dependent O.-U. Process

✗ Choice of Stochastic Differential Equation to built a stochastic process with given mean and variance

➩ "Ornstein-Uhlenbeck like" SDE - non-homogeneous in time

$$X_t = X_0 + \int_0^t \big(a(s)X_s + b(s)\big)ds + \int_0^t \sigma(s)dB_s$$

➩ Explicit form of X_t

$$X_t = \phi_0(t)\left[X_0 + \int_0^t \frac{b(s)}{\phi_0(s)}ds + \int_0^t \frac{\sigma(s)}{\phi_0(s)}dB_s\right]$$

where

$$\phi_u(t) \triangleq \exp\left(\int_u^t a(s)ds\right) \quad (u \leq t)$$

(Hypothesis on a and σ e.g. as $\int_0^{+\infty} a(u)du = -\infty$, $\int_0^{+\infty}\left(\frac{\sigma(u)}{\phi_0(u)}\right)^2 du = +\infty$)

Time dependent O.-U. Process

✗ From explicit expression of X_t or from Chapman-Kolmogorov equations:

- Mean

$$\mathbb{E}[X_t] = \phi_0(t)\left(\mathbb{E}[X_0] + \int_0^t \frac{b(s)}{\phi_0(s)}ds\right)$$

- Covariance

$$\mathbf{cov}(X_t, X_s) = \phi_0(t)\phi_0(s)\left\{\mathbf{var}[X_0] + \int_0^{t\wedge s}\left(\frac{\sigma(s)}{\phi_0(s)}\right)^2 ds\right\}$$

- Variance

$$\mathbf{var}\,[X_t] = \phi_0^2(t)\left\{\mathbf{var}[X_0] + \left(\int_0^t\left(\frac{\sigma(s)}{\phi_0(s)}\right)^2 ds\right)\right\}$$

Time dependent O.-U. Process

- Hypothesis: $\sigma(t) = \sigma$ is a constant
- **Choice of $a(t)$, $b(t)$ and σ such that:**

$$\mathbb{E}[X_t] = m(t) \text{ and } \mathbf{var}[X_t] = v(t)$$

where m and v are chosen (parametric) continuously differentiable functions, $\sigma > 0$ and $\forall t > 0$, $v(t) > 0$ and $\sigma^2 > v'(t)$.

$$\Rightarrow \; a(t) = \frac{v'(t) - \sigma^2}{2v(t)} \text{ and } b(t) = m'(t) - a(t)m(t)$$

$$X_t = X_0 + \int_0^t \left\{ \left(\frac{v'(s) - \sigma^2}{2v(s)} \right) X_s + m'(s) - m(s) \left(\frac{v'(s) - \sigma^2}{2v(s)} \right) \right\} ds + \sigma \int_0^t dB_s$$

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$$\Rightarrow a(t) = \frac{v'(t) - \sigma^2}{2v(t)} \text{ and } b(t) = m'(t) - a(t)m(t)$$

$$X_t = e^{-\alpha(t,0)}\left(X_0 - \beta(t,0) + \int_0^t \sigma e^{\alpha(u,s)} dB_u\right)$$

with $\alpha(t,s) = -\int_s^t a(u)du$ and $\beta(t,s) = m(s) - m(t)e^{\alpha(t,s)}$

Properties & Particular cases

✗ Mean Reverting property : long-term prognosis not influenced by last observation

$$\mathbb{E}[X_t|X_s = y] = m(t) + (y - m(s)) \,\mathbf{exp}\left(\int_s^t \frac{v'(u) - \sigma^2}{2v(u)} du\right)$$

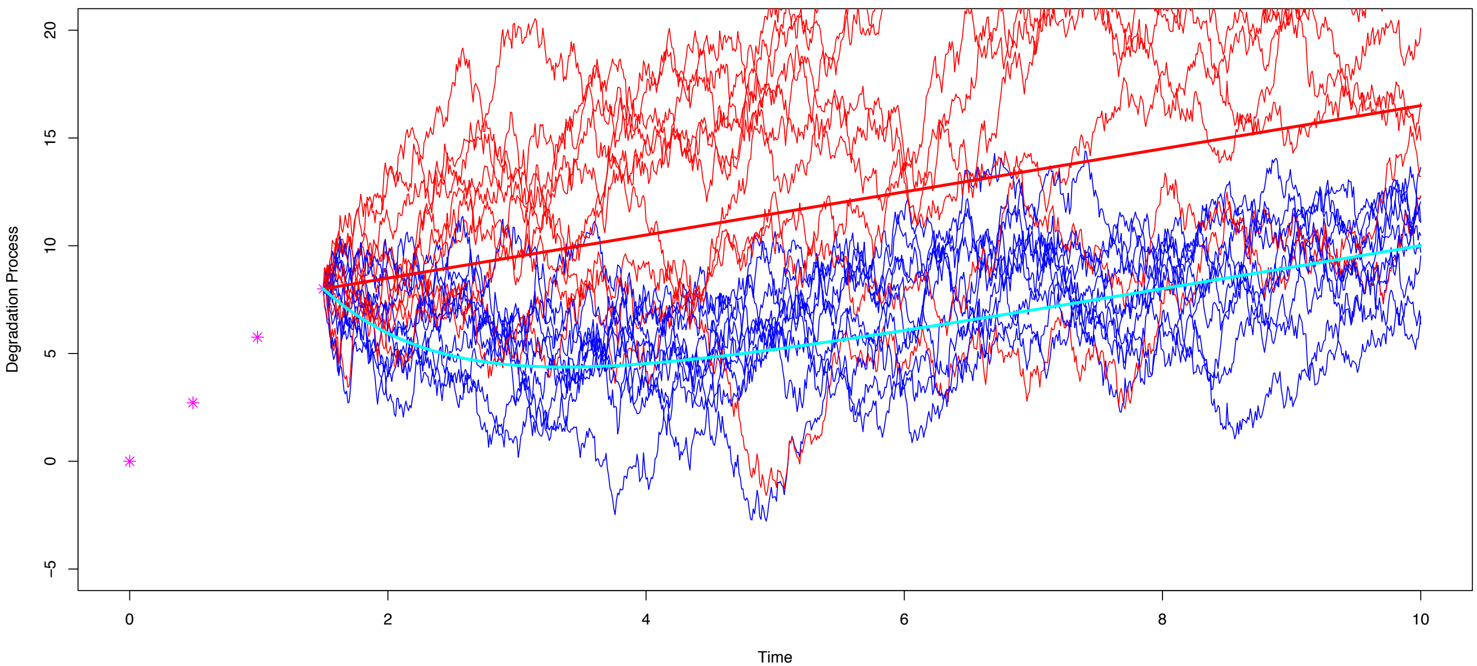

Fig.1 : Brownian and O.-U. processes with identical mean function starting from the origin

Properties & Particular cases

✗ Mean Reverting property : long-term prognosis not influenced by last observation

$$\mathbb{E}\left[X_t | X_s = y\right] = m(t) + \left(y - m(s)\right) \exp\left(\int_s^t \frac{v'(u) - \sigma^2}{2v(u)} du\right)$$

✗ Remarks

- If $v(t) = \sigma^2 t$ (i.e. $a(t) = 0$) then $X_t = m(t) + \sigma B_t$
- If $v(t) = v$ constant then $a(t)$ constant and $v = -\frac{\sigma^2}{2a}$

More generally if $a(t)$ is a constant (< 0) then

$$v(t) = \left(v(0) + \frac{\sigma^2}{2a}\right) e^{2at} - \frac{\sigma^2}{2a} \xrightarrow{t \to +\infty} -\frac{\sigma^2}{2a}$$

Maximum Likelihood Estimation

Estimation (MLE)

✗ Density function of X_t, with $p(x,t|y,s) = \mathscr{L}(X_t|X_s = y)$:

$$p(x,t|x_s,s) = \frac{e^{\alpha(t,s)}}{\sqrt{4\pi\gamma(t,s)}} \exp\left(-\frac{(xe^{\alpha(t,s)} + \beta(t,s) - y)^2}{4\gamma(t,s)}\right)$$

where $\alpha(t,s) = -\ln(\phi_s(t))$, $\beta(t,s) = m(s) - \frac{m(t)}{\phi_s(t)}$ and $\gamma(t,s) = \int_s^t \frac{\sigma^2}{2\phi_s^2(u)} du$.

✗ Proof (different possible ways):

- As a solution of the Fokker-Planck equation

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}\Big(a(t)x + m'(t) - a(t)m(t)\Big)p(x,t) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial^2 x}p(x,t)$$

Estimation (MLE)

✗ Density function of X_t, with $p(x, t|y, s) = \mathscr{L}(X_t | X_s = y)$:

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✗ Proof (different possible ways):

- From the explicit expression of X_t starting from y at time s

$$X_t = e^{-\alpha(t,s)}\left(y - \beta(t,s) + \int_s^t \sigma e^{\alpha(u,s)} dB_u\right)$$

Z_t $(t \geq s)$ zero mean Gaussian process $\sim \mathscr{N}(0, 2\gamma(t,s))$

Estimation (MLE)

✗ Data set: n components.
For component i, m_i records - $\{(t_{i,j}, x_{i,j}), j = 1, \dots, m_i\}$

✗ Log-likelihood:

$$\log L(\theta) = \sum_{i=1}^{n} \sum_{j=0}^{m_i-1} \log(p_\theta(x_{i,j+1}, t_{i,j+1} | x_{i,j}, t_{i,j}))$$

✗ Parameters: choice of parametric expressions of $m(t)$ and $v(t)$ depending on the data set

e.g. $m(t) = \lambda((t+1)^\mu - 1) + m_0$ and $v(t) = -\frac{\sigma^2}{2a}(1 - \exp(2at))$

⇨ $\theta = (\lambda, \mu, a, \sigma, m_0)$

Numerical results

- Log-likelihood maximization with classical numerical algorithm (Nelder-Mead)
- Illustration on partial data proposed by EDF for AMMSI

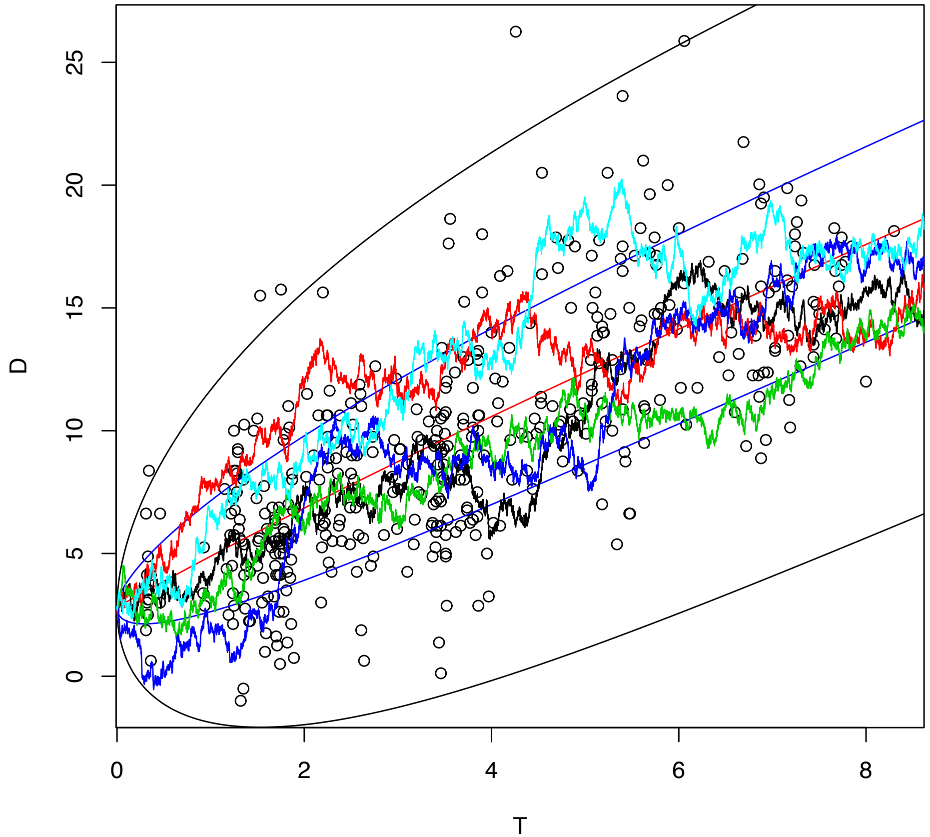
$\lambda = 2.44$

$\mu = 0.89$

$\sigma = 2.46$

$\alpha = -0.18$

$m_0 = 2.8$



Residual Useful Lifetime

Residual Useful Lifetime

- $\text{RUL}_s = \inf\{t \geq s, \quad X_t \notin \mathcal{U}\} - s$ where $\mathcal{U}$ = useful domain
  ➩ case of a constant failure limit: system failure as the degradation level reaches a given value $\mathcal{U} = \{x\,;\, x < L\}$

- First passage problem = first hitting time of the failure level L based on the degradation level at last inspection time
  ➩ Quantity of interest:

$$g(u|y,s) = \mathcal{L}(\text{RUL}_s | X_s = y)(u)$$

- Numerical expression of $g(u|y,s)$ as the solution of a non-singular Voltera integral equation

$$g(t|y,s) = -2K(L,t|y,s) + 2\int_s^t g(u|y,s)K(L,t|L,u)du.$$

Voltera integral equation

- Result established for Gauss-Markov process[1] (see *Buonocore et al.(1987)*)

$$Y_t = \eta(t) + h_2(t) B_{h_1(t)/h_2(t)}$$

➩ expression of the kernel as:

$$K(L,t|y,s) = \Big\{ -\frac{\eta'(t)}{2} - \frac{L-\eta(t)}{2} \frac{h_1'(t)h_2(s) - h_2'(t)h_1(s)}{h_1(t)h_2(s) - h_2(t)h_1(s)} - \frac{y-\eta(s)}{2} \frac{h_2'(t)h_1(t) - h_2(t)h_1'(t)}{h_1(t)h_2(s) - h_2(t)h_1(s)} \Big\} p(L,t|y,s)$$

- Time dependent O.-U. process can be written as

$$X_t = (y - \beta(t,s))e^{-\alpha(t,s)} + e^{-\alpha(t,s)} B_{2\gamma(t,s)}$$

➩ Kernel expression by taking $\eta(t) = (y - \beta(t,s))e^{-\alpha(t,s)}$, $h_1(t) = 2e^{-\alpha(t,s)}\gamma(t,s)$ and $h_2(t) = e^{-\alpha(t,s)}$ for $t \geq s$.

[1] B_* standard Brownian motion

First passage time

✗ Trapezoid numerical scheme:

$$g_1 = -2K(L(t_0+h), t_0+h | x_0, t_0)$$

$$g_k = -2K(L(t_0+kh), t_0+kh | x_0, t_0) + 2h\sum_{j=1}^{k-1} g_j K(L(t_0+kh), t_0+kh | L(t_0+jh), t_0+jh)$$

$$k = 2, 3, \ldots$$

✗ CDF of the first passage time ($t = 0, y = 0$)

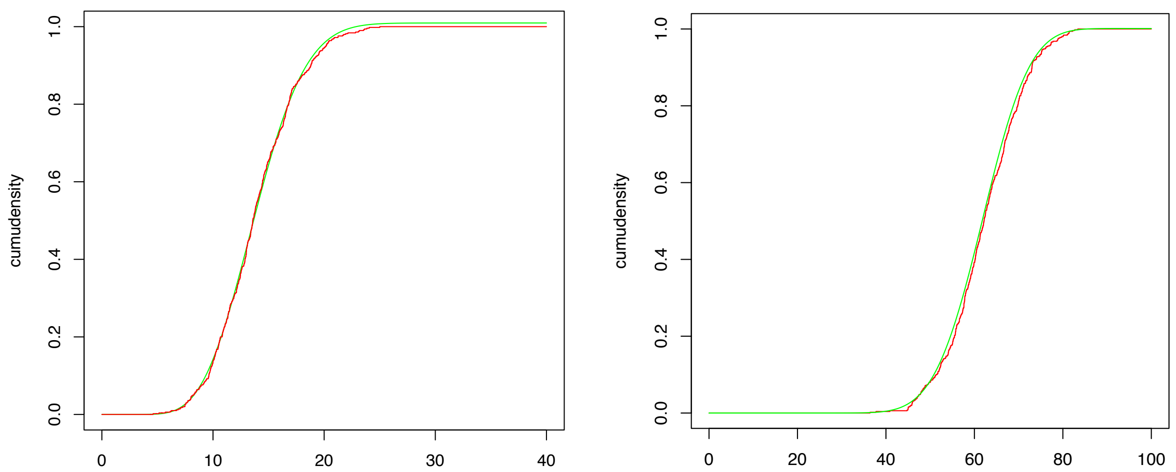


Fig2: cdf - Volterra (green) and Monte Carlo (red) for $L = 25$ (left) and $L = 50$ (right)

Linear diffusion and Time dependent O.-U. Comparison

Framework for comparison

- Data set: 415 records of degradation, 159 independent equipments
- Two possible processes with mean $m(t) = \lambda\big((t-1)^{\mu} - 1\big) + m_0$

($X_0 = m_0$ and $v_0 = 0$ given)

- Linear diffusion process: 4 parameters (λ, μ, m_0, σ)

$$X_t^{\text{BM}} = m(t) + \sigma B_t$$

- Time dependent O.-U. process: 5 parameters (λ, μ, m_0, σ, a)

$$X_t^{\text{OU}} = \int_0^t \Big(a\big(X_s - m(s)\big) + m'(s)\Big)ds + \int_0^t \sigma dB_s$$

- Maximum likelihood estimation

parameters	BM	OU
λ	1.8735	2.4403
μ	1.0059	0.8892
σ	2.1529	2.4641
a	–	-0.1807
m_0	2.9881	2.8075
variance	4.6352t	16.8033(1-exp(-0.3613t))

Exemples of trajectories

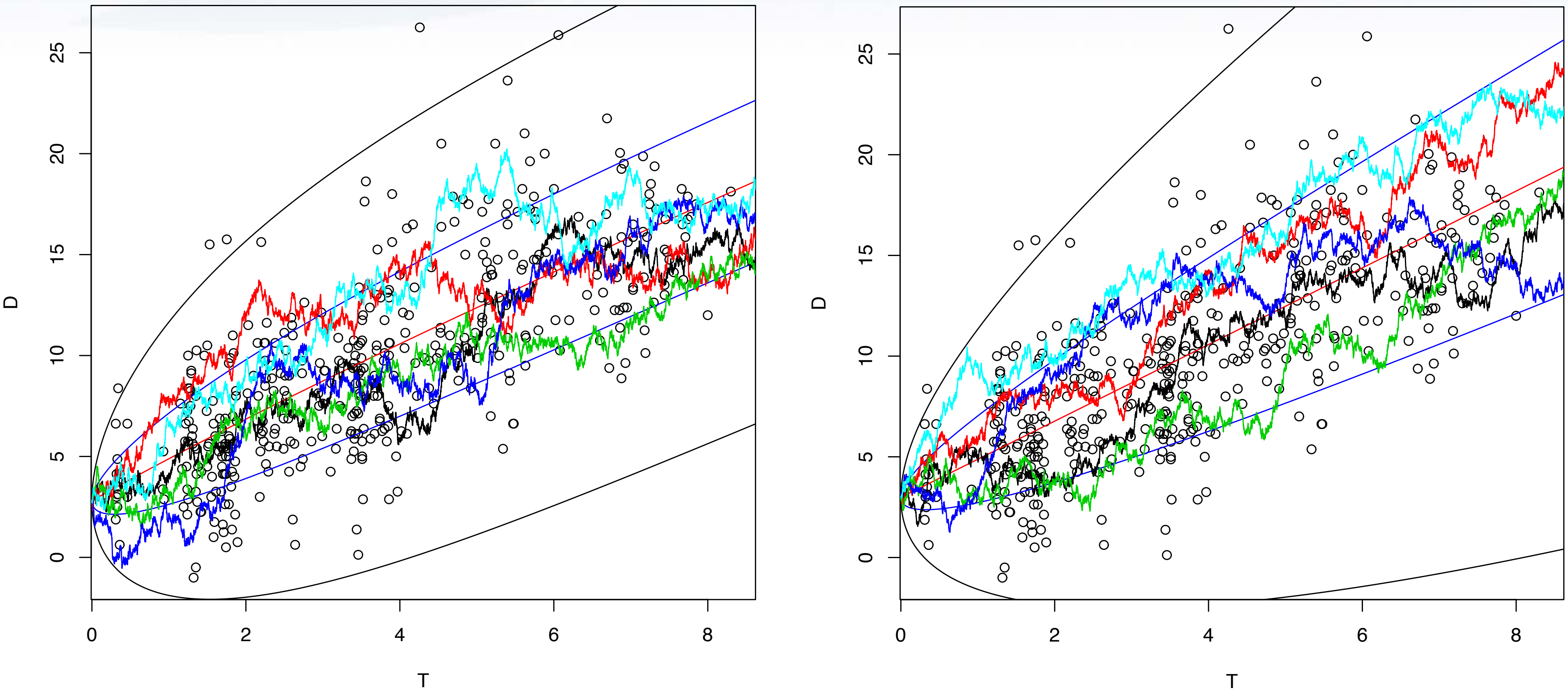


Fig.3: Exemple of trajectories for O.-U. process (left) and Brownian Motion (right)

Evaluation criterions and results

- Three criterions (heuristic - non i.d. random variables)

1. Akaïke Information Criterion AIC $= 2k - 2\ln L(\theta^*)$
2. $C_1 = \sum_{i=1}^{n}\sum_{j=0}^{m_i-1} P\Big(\big(X_{t_{i,(j+1)}} - x_{i,(j+1)}\big) \in [-\epsilon, \epsilon] | X_{t_{i,j}} = x_{i,j}\Big)$
3. $C_2 = \sum_{i=1}^{n}\sum_{j=0}^{m_i-1}\sum_{k=j+1}^{m_i-1} P\Big(\big(X_{t_{i,k}} - x_{i,k}\big) \in [-\epsilon, \epsilon] | X_{t_{i,j}} = x_{i,j}\Big)$

- Fitting goodness results

criterion[2]	BM	OU
AIC	2048.64	2021.044
C_1	115.0321	119.3473
C_2	193.2144	204.8864

[2] In the calculation of C_1 and C_2, the tolerance level is $\epsilon = 0.25$ (step-size = 0.01 for numeric integration).

First passage probability laws

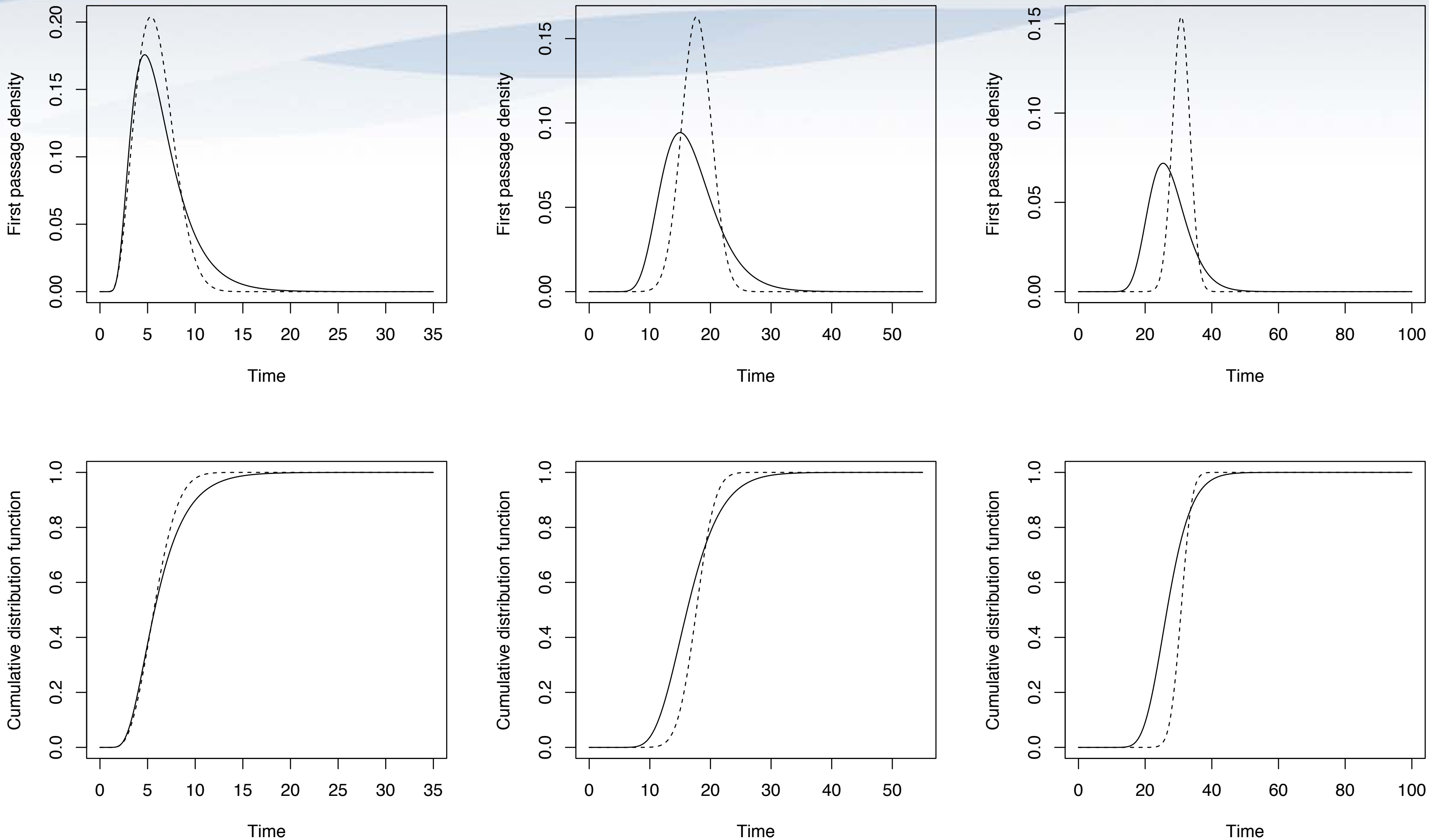


Fig.4: First passage pdf and cdf for $\mathrm{L} = 15, 35$ and 55

obtained with O.-U. process (dashed) and Brownian Motion (solid)

Conclusion

Conclusion

✗ Construction of a stochastic process for degradation modeling

- starting from Ornstein-Uhlenbeck process
- possible way to take into account some specific mean or variance shapes

✗ Numerical feasibility study for numerical estimation

- of the process parameters (parametric expression of mean and variance, constant diffusion coefficient)
- of the Remaining Useful Lifetime

... and still work to do...